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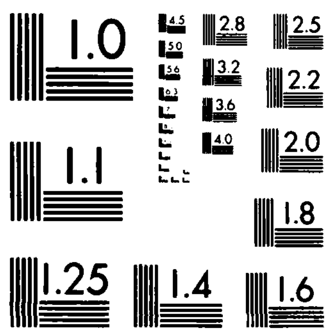

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Antistarshapedness, Dispersiveness and Mixtures

by

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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Abstract

Necessary and sufficient conditions are given for certain classes of distributions to be closed under mixtures. The relevant classes are defined in terms of antistarshapedness or dispersiveness. One of the main results characterizes distributions with log concave densities.

1. Introduction.

A basic tool in the analysis of univariate survival data is the quantile plot, $(G^{-1}(t), F_n^{-1}(t))$, where G is a fixed distribution of interest and F_n^{-1} is the empirical quantile function. For large sample sizes $F_n \approx F$, where F is the distribution of the observations. So the behavior of the empirical quantile plot should be similar to $(G^{-1}(t), F^{-1}(t))$ which is essentially the same as $(t, G^{-1}F(t))$. The study of this behavior leads quite naturally to classes of distributions $\{F: G^{-1}F$ has property $P\}$. Such classes were first studied by Van Zwet (1964) when P was the property of convexity.

Certain well-known classes fall within this framework. When G is exponential the IFR (increasing failure rate), IFRA (increasing failure rate average), DFR and DFRA are the classes where property P is, respectively, convexity, starshapedness, concavity and antistarshapedness. The latter two classes are known to be closed under mixtures (Barlow and Proschan, 1975, Section 4.4). For the arbitrary case necessary and sufficient conditions are given in Leon and Lynch (1983).

Corresponding to each lifetime random variable (r.v.) X with distribution F is a r.v. $\hat{X} = \log X$ with distribution $\hat{F}(x) = F(e^x)$, $-\infty < x < \infty$. A distribution $\hat{F}$ is said to be more <u>dispersed</u> than $\hat{G}$, written $\hat{G} \overset{d}{<} \hat{F}$, if

$$(1.1) \qquad \hat{G}^{-1}(\beta) - \hat{G}^{-1}(\alpha) \leq \hat{F}^{-1}(\beta) - \hat{F}^{-1}(\alpha) \qquad \alpha < \beta$$

This notion of dispersion is equivalent to $G^{-1}F$ being antistarshaped, i.e.,

$$(1.1') \quad \theta G^{-1}F(x) \le G^{-1}F(\theta x) \qquad 0 < \theta < 1,\ x > 0,$$

written $F \overset{a}{<} G$, and has been studied by a number of authors - Doksum (1969), Saunders and Moran (1978), Bickel and Lehmann (1979), Lewis and Thompson (1981) and Shaked (1982). Of particular note is Lewis and Thompson's result that the preservation of the dispersive ordering characterizes distributions with log concave densities.

In Section 2, we use the Lewis and Thompson characterization along with results in Leon and Lynch (1983) and the correspondence between (1.1) and (1.1') to find necessary and sufficient conditions on G for classes of distributions antistarshaped with respect to G to be closed under mixtures. Then using this same correspondence we obtain necessary and sufficient conditions for classes more dispersed than $\hat{G}$ to be closed under mixtures, one of which is that $\hat{G}$ have a log concave density.

2. The antistarshaped ordering and dispersiveness. In this section, the relationship between the antistarshaped ordering and the dispersive ordering is exploited to prove mixture results. The following definitions and lemma are needed. Throughout it is assumed that the support of the life distribution G is an interval $[0,c]$ if $c < \infty$ or $[0,\infty)$ if $c = \infty$.

Definition. A distribution $\hat{G}$ is said to be two point dispersive if $\hat{G} \overset{d}{<} \hat{G} * \hat{F}$ whenever $\hat{F}$ is a discrete distribution with only two mass points and where * denotes the convolution operation. The distribution $\hat{G}$ is said to be dispersive if $\hat{G} * \hat{F} \overset{d}{<} \hat{G} * \hat{H}$ whenever $\hat{F} \overset{d}{<} \hat{H}$.

Lemma 2.1. The following are equivalent:

(i) $\hat{G}$ is two point dispersive,

(ii) $\hat{G}$ is dispersive, and

(iii) $\hat{G}$ is absolutely continuous with a log concave density.

Proof (ii) $\leftrightarrow$ (iii). This is in Lewis and Thompson (1981).

(ii) $\rightarrow$ (i). This is obvious since it is easy to see that any distribution is more dispersed than a degenerate one.

(i) $\rightarrow$ (iii). If $\hat{G}$ is twice differentiable the argument in the proof of the only if part of Lewis and Thompson (1981) shows that $\hat{G}$ has a log concave density. The proof then follows as in the last paragraph of Lynch, Mimmack and Proschan (1983). □

Definition. The hazard transform for a life distribution G is given by

$$T_p(u,v) = G^{-1}(pG(u)+\bar{p}G(v))$$

where $0 < p < 1$, $\bar{p} = 1 - p$, and $0 \leq u, v < \infty$.

The following notation is needed. For two life distributions we denote the scale mixture $\int G(\lambda^{-1}x)dF(\lambda)$ by G_F.

Let

$$\mathcal{A}_G = \{F: F \overset{a}{<} G\} \text{ and } \mathcal{D}_{\hat{G}} = \{\hat{F}: \hat{G} \overset{d}{<} \hat{F}\}.$$

Thus $\mathcal{D}_{\hat{G}} = \{\hat{F}: F \epsilon \mathcal{A}_G\}$ by the equivalence of (1.1) and (1.1').

The main results are given in the next two theorems.

Theorem 2.2. The following are equivalent:

(i) $\mathcal{A}_G$ is closed under mixtures,

(ii) $T_p(u,v)$ is antistarshaped for each p,

(iii) $\hat{G}$ is dispersive (and so, $\hat{G}$ has a log concave density), and

(iv) $\mathcal{D}_{\hat{G}}$ is closed under mixtures.

Proof (i) $\leftrightarrow$ (ii). This is just a restatement of Theorem 2.4(a) of Leon and Lynch (1983).

(ii) $\leftrightarrow$ (iii). Because of Lemma 2.1, it suffices to show that (ii) is equivalent to $\hat{G}$ being two point dispersive. Let $\hat{F} = p\delta_x + \bar{p}\delta_y$ where $\delta_x(z) = 0$ if $x \neq z$ and $= 1$ if $x = z$. Let $u = e^x$ and $v = e^y$. Then $F = p\delta_u + \bar{p}\delta_v$. Thus,

$$
\begin{aligned}
\hat{G} \overset{d}{<} \hat{G} * \hat{F} \text{ iff } & G_F \overset{a}{<} G \\
\text{iff } & G^{-1}G_F(\theta z) \geq G^{-1}G_F(z) \text{ for } \theta \in (0,1) \\
\text{iff } & T_p(uz, uz) \text{ is antistarshaped} \\
\text{iff } & T_p(u,v) \text{ is antistarshaped.}
\end{aligned}
$$

(i) $\leftrightarrow$ (iv). Ignoring certain measure theory details which are easily resolved and left to the reader, for a measure μ on A_G, let $\hat{\mu} = \mu I^{-1}$ where I: $A_G \to \mathcal{D}_{\hat{G}}$ is the 1-1 mapping $I(F) = \hat{F}$. Then,

$$\hat{G} \overset{d}{<} \int \hat{H} d\hat{\mu} \text{ iff } \int H d\mu \overset{a}{<} G.$$

Thus A_G is closed under mixtures if and only if $\mathcal{D}_G$ is closed under mixtures. □

As an immediate consequence of Lemma 2.1 and the equivalence of (1.1) and (1.1') we have

Theorem 2.3. $G_F \overset{a}{<} G_H$ whenever $F \overset{a}{<} H$ if and only if $\hat{G}$ is dispersive.

Corollary 2.4. Let $\hat{G}$ be dispersive. If $F \overset{a}{<} H$, then $G_F - G_H$ has at most one sign change and if one occurs it is from $-$ to $+$. If, in addition, $\int \lambda dF(\lambda) = \int \lambda dH(\lambda)$, then $G_F - G_H$ has exactly one sign change.

Proof. The proof is the same as for the DFRA case, i.e., G is the exponential distribution which is in Barlow and Proschan (1975). It is reproduced here for completeness.

Since $G_F \overset{a}{<} G_H$ by Theorem 2.3, $G_F - G_H$ has at most one sign change and it is from - to + if one occurs. If $\int \lambda dF(\lambda) = \int \lambda dH(\lambda)$, then $\int x dG_F(x) = \int x dG_H(x)$. Thus, $G_F - G_H$ must have at least one sign change. □

Remark. Corollary 2.4 should be contrasted with Theorem 4 of Shaked (1980) where the ordering on the mixing distributions and the mixed distributions is the dilation ordering and with Theorem 1 of the same paper.

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